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COMMENT

1/*N* expansion and the uncertainty principle

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Abstract. The leading-order term of the $1/N$ expansion is derived in a new way and is thus given a simple physical interpretation.

In recent years the semiclassical $1/N$ expansion has become a subject of extensive investigations (Mlodinow and Papanicolaou 1980, 1981, Mlodinow and Shatz 1982, Van Der Merwe 1983, Moreno and Zepeda 1984, and references therein) mainly for two reasons. Firstly, the method is non-perturbative and hence can be applied to strong coupling cases ('t Hooft 1974a, b, Witten 1979a), and secondly and perhaps more importantly, the very reason why such an expansion really works is not well understood and therefore calls for a critical examination of its mathematical foundations (Mlodinow and Papanicolaou 1980, 1981, Papanicolaou 1981). In this communication we give a new and simple derivation of the leading-order term of the $1/N$ expansion which leads to a simple physical interpretation of this term.

In $1/N$ expansion one starts with the radial part of the N-dimensional Schrödinger equation (Mlodinow and Shatz 1982)

$$\left[-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{l(l+N-2)}{2r^2}+V(r)\right]R(r)=ER(r), \tag{1}$$

which on substituting $R(r)=r^{-(N-1)/2}U(r)$, reduces to

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}U(r)+k^2\left(\frac{(1-1/k)(1-3/k)}{8r^2}+\tilde{V}(r)\right)U(r)=EU(r), \tag{2}$$

where $\tilde{V}(r)=V(r)/k^2$, $k=N+2l$, and the units have been chosen so that $m=\hbar=1$. In this comment we shall be interested only in the ground state of the system for which $k=N$. In the limit $N\to\infty$, the most significant contribution to the ground state energy is given by

$$E_\infty=N^2(\tfrac{1}{8}r_0^{-2}+\tilde{V}(r_0)), \tag{3}$$

where r_0 is the value of r for which the effective potential $(\frac{1}{8}r^{-2}+\tilde{V}(r))$ is minimum. E_∞ is the zeroth-order result of the $1/N$ expansion. Quantum fluctuations around the classical minimum r_0 can however be incorporated in the higher-order corrections (Mlodinow and Shatz 1982).

It is well known that the leading term of the $1/N$ expansion does not in general yield satisfactory results (Witten 1979b, Mlodinow and Papanicolaou 1981). Mlodinow and Papanicolaou (1981) have rightly pointed out that this is not surprising in view of the semiclassical nature of the zeroth-order approximation in the $1/N$ expansion. In what follows we derive the leading-order term of the $1/N$ expansion in an alternative way which, we believe, allows us to state more conclusively that the leading-order approximation may not be sufficient in general.

The energy of a particle in N dimensions is given classically by

$$E = \tfrac{1}{2}p_{x_1}^2 + \tfrac{1}{2}p_{x_2}^2 + \ldots + \tfrac{1}{2}p_{x_N}^2 + V(r), \tag{4}$$

where $p_{x_1}, p_{x_2}, \ldots, p_{x_N}$ are the Cartesian components of the linear momentum of the particle and $V(r)$ is the potential depending only on the radial coordinate r.

Now taking the uncertainty product of the system to be minimum so that $\Delta x \Delta p_x = \frac{1}{2}(\hbar = 1)$, and assuming that $(\Delta x)^2 \sim x^2$ and $(\Delta p_x)^2 \sim p_x^2$, one can write (4) as

$$E \simeq \tfrac{1}{8}x^{-2} + \tfrac{1}{8}x^{-2} + \ldots + \tfrac{1}{8}x_N^{-2} + V[(x_1^2 + x_2^2 + \ldots + x_N^2)^{1/2}]. \tag{5}$$

If one further assumes that the spreads in the position coordinates along different axes are also of the same order, then (4) finally becomes

$$E \simeq \tfrac{1}{8}N^2 r^{-2} + V(r),$$

which when minimised with respect to r should give the ground state energy of the system and becomes exactly identical to (3). Hence the leading-order result of the semiclassical non-perturbative $1/N$ expansion is simply an order of magnitude estimate of the ground state energy assuming the system's uncertainty product to be a minimum.

It is interesting to recall in this context that the $1/N$ expansion when applied to a quantum mechanical simple harmonic oscillator gives the exact result in the zeroth order, each higher-order correction being identically zero. Now it is clear from the above derivation that this is not merely coincidental, but is an immediate consequence of the fact that the ground state wavefunction of a simple harmonic oscillator is a Gaussian wave for which the uncertainty is indeed minimum. Obviously then for most systems the zeroth-order approximation is not enough and one will have to consider the higher-order corrections in order to take care of the deviation from the minimum uncertainty.

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